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A nested recurrence relation with Fibonacci values at Fibonacci indices

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Define

$$a(n) = \begin{cases} 0 & \text{if } n \leq 0 \\ n - 1 - a(n-1) - a(a(n-2)) - a(a(a(n-3))) - a(a(a(a(n-4)))) - \cdots & \text{otherwise} \end{cases}$$

Below is a table of values of this sequence and further values may be found in the OEIS in sequence

$a(n)$	0	**1**	**1**	2	**2**	2	3	**3**	4	4	4	4	**5**	5	6	6	7	7	7	8	**8**	9
n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22

It is known that

$$a(F_{2m}) = F_{2m-2},$$

where F_n is the n-th Fibonacci number.

The open problem is to show also that

$$a(F_{2m+1}) = F_{2m-1}.$$

One could also ask for a "closed form" for $a(n)$.

This problem is taken from: *Morphic Words and Nested Recurrence Relations*, by Marcel Celaya, Frank Ruskey, arXiv:1307.0153.

Note #1: The value of $a(n)$ is not the same as the value obtained by shifting the Zekendorf representation of n by two positions.

Note #2: There is an alternate interpretation in terms of the morphism σ below:

$$r \to r0,\ 0 \to 10,\ 1 \to 200,\ 2 \to 3000,\ 3 \to 40000,\ \ldots.$$

Here $a(n)$ is the number of non-zero characters in the length n prefix of the resulting infinite morphic word $r^{-1}\sigma^{\infty}(r)$. This interpretation may be found in the arXiv paper referred to above.