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Graphs with no equal length cycles

Chunhui Lai
Minnan Normal University, CHINA
laichunhui@mnnu.edu.cn

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Let $f(n)$ be the maximum number of edges in a graph on n vertices in which no two cycles have the same length. In 1975, Erdös raised the problem of determining $f(n)$ (see J.A. Bondy and U.S.R. Murty [1], p.247, Problem 11). Y. Shi [12] proved that

$$f(n) \geq n + [(\sqrt{8n-23}+1)/2]$$

for $n \geq 3$. E. Boros, Y. Caro, Z. Füredi and R. Yuster [2] proved that

$$f(n) \leq n + 1.98\sqrt{n}(1+o(1)).$$

Lai [7] proved that

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{40}{99}}.$$

and conjectured that

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} > \sqrt{2.444}.$$

Lai [6] proposed the following conjecture:

Conjecture [6].

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \leq \sqrt{3}.$$

It would be nice to prove that

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \leq \sqrt{3+\frac{3}{5}}.$$

Let $f_2(n)$ be the maximum number of edges in a 2-connected graph on n vertices in which no two cycles have the same length.

In 1988, Shi[12] proved that

For every integer $n \geq 3$, $f_2(n) \leq n + [\frac{1}{2}(\sqrt{8n-15}-3)]$.

In 1998, Guantao Chen, Jeno Lehel, Michael S.Jacobson, and Warren E. Shreve [3] proved that $f_2(n) \geq n + \sqrt{n/2} - o(\sqrt{n})$

In 2001, E. Boros, Y. Caro, Z. Füredi and R. Yuster [2] improved this lower bound significantly: $f_2(n) \geq n + \sqrt{n} - O(n^{\frac{9}{20}})$.

and conjectured that

$\lim \frac{f_2(n)-n}{\sqrt{n}} = 1$.

It is easy to see that this Conjecture implies the (difficult) upper bound in the Erdos Turan Theorem [4][5](see [2]).

Markström [11] raised the problem of determining the maximum number of edges in a hamiltonian graph on n vertices with no repeated cycle lengths.

Let $g(n)$ be the maximum number edges in an n-vertex, Hamiltonian graph with no repeated cycle length. J. Lee, C. Timmons [9] prove the following.

If q is a power of a prime and $n = q^2 + q + 1$, then

$$g(n) \geq n + \sqrt{n - 3/4} - 3/2$$

A simple counting argument shows that $g(n) < n + \sqrt{2n} + 1$.

It would be nice to determining $g(n)$ for infinitely many n.

J. Ma, T. Yang [10] prove a conjecture of Boros, Caro, Füredi and Yuster on the maximum number of edges in a 2-connected graph without repeated cycle lengths, which is a restricted version of a longstanding problem of Erdős. Their proof together with the matched lower bound construction of Boros, Caro, Füredi and Yuster show that this problem can be conceptually reduced to the seminal problem of finding the maximum Sidon sequences in number theory.

Any n-vertex 2-connected graph with no two cycles of the same length contains at most $n + \sqrt{n} + o(\sqrt{n})$ edges.

The survey article on this problem can be found in [8].

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